

ON THE WEYL AND RICCI TENSORS OF GENERALIZED ROBERTSON-WALKER SPACE-TIMES

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ABSTRACT. We prove theorems about the Ricci and the Weyl tensors on generalized Robertson-Walker space-times of dimension $n \geq 3$. In particular, we show that the concircular vector introduced by Chen decomposes the Ricci tensor as a perfect fluid term plus a term linear in the contracted Weyl tensor. The Weyl tensor is harmonic if and only if it is annihilated by Chen's vector, and any of the two conditions is necessary and sufficient for the GRW space-time to be a quasi-Einstein (perfect fluid) manifold. Finally, the general structure of the Riemann tensor for Robertson-Walker space-times is given, in terms of Chen's vector. In $n = 4$ a GRW space-time with harmonic Weyl tensor is a Robertson-Walker space-time.

1. INTRODUCTION

The beautiful theorem by Chen (2014, [1]) gives a simple and covariant characterisation of Generalized Robertson-Walker space-times (GRW). It states that a Lorentzian manifold of dimension $n \geq 3$ is a GRW if and only if there exists a vector field X_j , which we name *Chen's vector*, such that $X^j X_j < 0$ and

$$(1) \quad \nabla_j X_k = \rho g_{jk}$$

where ρ is some scalar field. The equation is the defining property of a concircular vector, introduced by Fialkow [2] (Yano gave the same name to a broader class of vectors [3]). The existence of such a vector field is a necessary and sufficient condition for a local expression of the metric with the warped form

$$(2) \quad ds^2 = -dt^2 + q^2(t) g_{\mu\nu}^*(\vec{x}) dx^\mu dx^\nu$$

where $g_{\mu\nu}^*$ is the metric tensor of a Riemannian submanifold M^* parametrised by \vec{x} . The local form (2) is actually how GRW manifolds were defined in 1995 by Alias et al. [4], and studied subsequently (see for example [5, 6, 7]).

The strong property of Chen's vector upon differentiation allows for a determination of several interesting properties of GRW manifolds. The following two sections are devoted to the Ricci and to the Weyl tensor respectively. After showing that Chen's vector is an eigenvector of the Ricci tensor, the Ricci tensor is expressed as the sum of a perfect fluid term and a term proportional to the Weyl tensor.

The main theorem states that the Weyl tensor is harmonic ($\nabla^m C_{jklm} = 0$) if and only if $C_{jklm} X^m = 0$. Any of the two conditions is necessary and sufficient for the GRW manifold to be a quasi-Einstein (or perfect fluid) space-time, i.e. for the Ricci tensor to have the form $R_{ij} = A g_{ij} + B u_i u_j$.

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Part of these statements existed in the literature: in 1994 Gebarowski [8] proved that the fibers of a GRW space-time are Einstein (i.e. the Ricci tensor of the submanifold M^* in the decomposition (2) has the form $R_{\mu\nu}^* = R^* g_{\mu\nu}/(n-1)$) if and only if $\nabla^m C_{jklm} = 0$. Then, Sanchez (1999, [9]) stated that the fibers of a GRW space-time are Einstein if and only if the GRW manifold is quasi-Einstein (or perfect-fluid). Recently, Mantica et al. (2016, [10]) showed the converse: a perfect-fluid Lorentzian manifold with $\nabla^m C_{jklm} = 0$ is a GRW space-time (see also [11]). In this presentation, besides other results, we show that the same propositions hold with the algebraic condition $C_{jklm} X^m = 0$ where X is Chen's vector. In the last section we consider conformally flat GRW space-times, i.e. Robertson-Walker space-times, and provide the general form of the Riemann tensor. In particular, the aforementioned results imply that in $n = 4$ a conformally harmonic GRW space-time is a Robertson-Walker space-time.

2. THE RICCI TENSOR

Theorem 2.1. *On a GRW manifold:*

1) *Chen's vector X_j is an eigenvector of the Ricci tensor,*

$$(3) \quad R_{jm} X^m = \xi X_j,$$

and the following holds for the eigenvalue:

$$(4) \quad \xi = -(n-1) \frac{X^k \nabla_k \rho}{X^2}, \quad \nabla_i \xi = \theta X_i$$

where θ is a scalar field.

2) *the Ricci tensor can be expressed in terms of the Weyl tensor, the curvature scalar R , the eigenvalue ξ and Chen's vector:*

$$(5) \quad R_{kl} = (n-2) C_{jklm} \frac{X^j X^m}{X^2} + \frac{R - \xi}{n-1} \left(g_{kl} - \frac{X_k X_l}{X^2} \right) + \xi \frac{X_k X_l}{X^2};$$

Proof. By Chen's theorem, on a GRW manifold there is a vector field X_j such that $\nabla_i X_j = \rho g_{ij}$. Then $[\nabla_i, \nabla_j] X_k = (\nabla_i \rho) g_{jk} - (\nabla_j \rho) g_{ik}$ i.e. $R_{ijk}^m X_m = (\nabla_i \rho) g_{jk} - (\nabla_j \rho) g_{ik}$. A contraction with g^{jk} gives $R_{im} X^m = -(n-1) \nabla_i \rho$, while the contraction with X^k gives $0 = X_j \nabla_i \rho - X_i \nabla_j \rho$, then $X^2 \nabla_j \rho = X_j (X^i \nabla_i \rho)$ i.e. $\nabla_i \rho$ is proportional to X_i . Therefore (3) follows.

Moreover:

$$(6) \quad R_{ijk}^m X_m = -\frac{\xi}{n-1} (X_i g_{jk} - X_j g_{ik})$$

The derivative of (6) is

$$X^m \nabla_s R_{ijkm} + \rho R_{ijks} = -\frac{\nabla_s \xi}{n-1} (X_i g_{jk} - X_j g_{ik}) - \frac{\rho \xi}{n-1} (g_{si} g_{jk} - g_{ik} g_{js}),$$

the sum on cyclic permutations of indices $si j$ and the second Bianchi identity give:

$$0 = g_{jk} (X_i \nabla_s \xi - X_s \nabla_i \xi) + g_{ik} (X_s \nabla_j \xi - X_j \nabla_s \xi) + g_{sk} (X_j \nabla_i \xi - X_i \nabla_j \xi).$$

The contraction with g^{sk} finally gives $X_j \nabla_i \xi - X_i \nabla_j \xi = 0$, with solution $\nabla_i \xi = \theta X_i$, thus proving (4).

By definition, the Weyl tensor is

$$C_{jklm} = R_{jklm} + \frac{1}{n-2} (g_{jm} R_{kl} - g_{km} R_{jl} + R_{jm} g_{kl} - R_{km} g_{jl}) - R \frac{g_{jm} g_{kl} - g_{jl} g_{km}}{(n-1)(n-2)}$$

A contraction with X^m and (6) give:

$$(7) \quad C_{jkl}^m X_m = \frac{\xi - R}{(n-1)(n-2)} (X_j g_{kl} - X_k g_{jl}) + \frac{1}{n-2} (X_j R_{kl} - X_k R_{jl}).$$

Another contraction with X^j gives the Ricci tensor (5). \square

A covariant derivative of the eigenvalue equation (3), and use of (1) and $\nabla^k R_{kj} = \frac{1}{2} \nabla_j R$, where R is the curvature scalar, gives a relation that will be important in the sequel:

$$(8) \quad \frac{1}{2} X^k \nabla_k R = n\rho\xi - \rho R + X^2\theta$$

Multiplication of (6) by X_l and summation on cyclic permutations of ijl shows that Chen's vector is "Riemann compatible" [12]: $X_i X^m R_{jlk m} + X_j X^m R_{lik m} + X_l X^m R_{ij k m} = 0$. In general, Riemann implies Weyl compatibility [13]. In the present case (it can be checked with (7)):

$$(9) \quad X_i X^m C_{jkl m} + X_j X^m C_{kil m} + X_k X^m C_{ijl m} = 0.$$

3. THE WEYL TENSOR

We now focus on the Weyl tensor. It is useful to introduce the auxiliary symmetric trace-less tensor

$$C_{jk} = C_{ajkb} \frac{X^a X^b}{X^2}$$

Note the properties $X^j C_{jk} = 0$ and $X^j \nabla_l C_{jk} = -(\nabla_l X^j) C_{jk} = -\rho C_{kl}$, that will be frequently used. The contraction of (9) by X^i gives:

Proposition 3.1. *The Weyl tensor of a GRW manifold satisfies the identity:*

$$(10) \quad C_{jklm} X^m = X_j C_{kl} - X_k C_{jl}$$

It implies that $C_{jklm} X^m = 0$ if and only if $C_{kl} = 0$.

The general expression for the covariant divergence of the Weyl tensor is:

$$(11) \quad \nabla^m C_{jklm} = -\frac{n-3}{n-2} \left[\nabla_j R_{kl} - \nabla_k R_{jl} - \frac{g_{kl} \nabla_j R - g_{jl} \nabla_k R}{2(n-1)} \right].$$

We look for an expression in terms of the contracted tensor C_{jk} . The following covariant derivatives are evaluated with (1) and $\nabla_j \xi = \theta X_j$:

$$(12) \quad \begin{aligned} \nabla_j R_{kl} - \nabla_k R_{jl} &= (n-2)(\nabla_j C_{kl} - \nabla_k C_{jl}) \\ &+ \frac{1}{n-1} (n\rho\xi - \rho R + X^2\theta) \left(\frac{X_k}{X^2} g_{jl} - \frac{X_j}{X^2} g_{kl} \right) \\ &+ \frac{1}{n-1} \left[\left(g_{kl} - \frac{X_k X_l}{X^2} \right) \nabla_j R - \left(g_{jl} - \frac{X_j X_l}{X^2} \right) \nabla_k R \right] \\ &= \frac{1}{n-1} \left[\left(g_{kl} - \frac{X_k X_l}{X^2} \right) \nabla_j R - \left(g_{jl} - \frac{X_j X_l}{X^2} \right) \nabla_k R \right. \\ &\left. + \frac{1}{2} \left(\frac{X_k X_s}{X^2} g_{jl} - \frac{X_j X_s}{X^2} g_{kl} \right) \nabla^s R \right] + (n-2)(\nabla_j C_{kl} - \nabla_k C_{jl}) \end{aligned}$$

because of the identity (8). Eq.(11) becomes:

$$(13) \quad \begin{aligned} \nabla^m C_{jklm} = & -(n-3)(\nabla_j C_{kl} - \nabla_k C_{jl}) + \frac{n-3}{(n-1)(n-2)} \frac{X_l}{X^2} (X_k \nabla_j R - X_j \nabla_k R) \\ & - \frac{n-3}{2(n-1)(n-2)} \left[g_{kl} \left(g_{jm} - \frac{X_j X_m}{X^2} \right) \nabla^m R - g_{jl} \left(g_{km} - \frac{X_k X_m}{X^2} \right) \nabla^m R \right] \end{aligned}$$

Its contraction with g^{kl} gives

$$(14) \quad \nabla^k C_{jk} = \frac{n-3}{2(n-1)(n-2)} \left(\nabla_j R - \frac{X_j X^l}{X^2} \nabla_l R \right).$$

Lemma 3.2.

$$(15) \quad X^j \nabla_j C_{kl} = -2\rho C_{kl}$$

Proof. The contraction of (13) with X^j is:

$$X^j \nabla^m C_{jklm} = -(n-3)(X^j \nabla_j C_{kl} + \rho C_{kl}) - \frac{(n-3)X_l}{2(n-1)(n-2)} \left(\nabla_k R - \frac{X_k X^j}{X^2} \nabla_j R \right)$$

With the aid of (14) and with a permutation of indices, it becomes:

$$(16) \quad \nabla^j (C_{jlk m} X^m) = -(n-3)(X^j \nabla_j C_{kl} + \rho C_{kl}) - X_l \nabla^m C_{km}$$

The left-hand-side of this equation is evaluated by means of (10): $\nabla^j (C_{jlk m} X^m) = \nabla^j (X_j C_{lk} - X_l C_{jk})$ i.e.

$$(17) \quad \nabla^j (C_{jlk m} X^m) = (n-1)\rho C_{kl} + X^j \nabla_j C_{kl} - X_l \nabla^j C_{jk}$$

The two equations imply (15). \square

The following statement is important:

Proposition 3.3. *If $X^l \nabla^m C_{jklm} = 0$ then:*

$$(18) \quad \nabla_i R = X_i \frac{X^m \nabla_m}{X^2} R$$

$$(19) \quad \nabla^m C_{jklm} = -(n-3)(\nabla_j C_{kl} - \nabla_k C_{jl})$$

Proof. Recall that $X^l \nabla_j C_{kl} = -\rho C_{jk}$. The contraction of (13) with X^l gives

$$(20) \quad X^l \nabla^m C_{jklm} = \frac{n-3}{2(n-1)(n-2)} (X_k \nabla_j R - X_j \nabla_k R)$$

If $X^l \nabla^m C_{jklm} = 0$ then $X_k \nabla_j R = X_j \nabla_k R$, with solution (18). Eq.(13) greatly simplifies and reduces to (19). \square

Since $X^l \nabla^m C_{jklm} = \nabla^m (C_{jklm} X^l)$, the proposition holds in particular if $\nabla^m C_{jklm} = 0$ or if $X^m C_{jklm} = 0$.

We are ready to prove the main theorem:

Theorem 3.4. *On a GRW space-time with Chen vector X_j*

$$(21) \quad C_{jklm} X^m = 0 \iff \nabla^m C_{jklm} = 0$$

Proof. If $C_{jklm}X^m = 0$ then $C_{jk} = 0$. The right-hand-side of (19) is zero, and the Weyl tensor is harmonic.

If $\nabla^m C_{jklm} = 0$ then $\nabla_j C_{kl} - \nabla_k C_{jl} = 0$. In particular:

$$X^j \nabla_j C_{kl} = -\rho C_{kl}$$

Because of (15) it is $C_{kl} = 0$ i.e. $C_{jklm}X^m = 0$. \square

Proposition 3.5. *On a GRW space-time with Chen vector X_k , $C_{jklm}X^m = 0$ if and only if*

$$(22) \quad R_{jk} = \alpha g_{jk} + \beta \frac{X_j X_k}{X^2}$$

for suitable scalars α and β .

Proof. If $C_{jklm}X^m = 0$, then (5) gives R_{jk} the perfect fluid form. If $R_{jk} = \alpha g_{jk} + \beta \frac{X_j X_k}{X^2}$ then $R = n\alpha + \beta$ and $\xi = \alpha + \beta$, and the right hand side of (7) is zero. \square

Because of the main theorem 3.4 we also have:

Proposition 3.6. *On a GRW space-time, it is $\nabla^m C_{jklm} = 0$ if and only if the Ricci tensor has the structure (22) (i.e. the manifold is quasi-Einstein).*

We end the section with a remark: the Ricci tensor of a GRW space-time with $\nabla_m C_{jkl}^m = 0$ has the property

$$(23) \quad [\nabla_i, \nabla_j]R_{kl} = -\frac{\xi}{n-1} [g_{jk}R_{li} - g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk}]$$

which defines a Ricci pseudo-symmetric manifold. This is shown by introducing the structure (22) in the evaluation

$$[\nabla_i, \nabla_j]R_{kl} = R_{ijk}^m R_{ml} + R_{ijl}^m R_{km} = \beta R_{ijk}^m X_m \frac{X_l}{X^2} + \beta R_{ijl}^m X_m \frac{X_k}{X^2}$$

With the aid of (6), relation (23) is obtained. This is in accordance with a result by Deszcz: a warped manifold with $ds^2 = \pm(dx^0)^2 + F(x^0)\tilde{g}_{\mu\nu}(\vec{x})dx^\mu dx^\nu$, where \tilde{g} is the metric tensor of a pseudo-Riemannian Einstein manifold, is Ricci pseudo-symmetric ([15], corollary 3.2; the result is also reported in [16]).

4. ROBERTSON-WALKER SPACE-TIMES

Robertson-Walker space-times are an important subclass of GRW space-times; they share the property of being conformally flat, $C_{jklm} = 0$. Let us then consider GRW space-times that are conformally flat.

The following theorem applies [17]: A GRW space-time is conformally flat if and only if the GRW manifold is the ordinary Robertson-Walker space-time (or: if and only if the submanifold M^* in the warped product (2) is a space of constant curvature).

With $C_{jklm} = 0$ the Ricci tensor has the structure (22) and the Riemann tensor is largely determined:

$$(24) \quad R_{jklm} = \frac{2\xi - R}{(n-1)(n-2)} (g_{kl}g_{jm} - g_{km}g_{jl}) + \frac{R - n\xi}{(n-1)(n-2)} \left[g_{jm} \frac{X_k X_l}{X^2} - g_{km} \frac{X_j X_l}{X^2} + g_{kl} \frac{X_j X_m}{X^2} - g_{jl} \frac{X_k X_m}{X^2} \right]$$

where X is Chen's vector, R is the curvature scalar, ξ is the eigenvalue of the Ricci tensor with eigenvector X .

Eq. (24) is the general form of the Riemann tensor of a Robertson-Walker space-time. The form characterises manifolds of quasi-constant curvature, introduced by Chen and Yano in 1972, [18].

In four-dimensions, the Weyl tensor on a pseudo-Riemannian manifold has a special property: if $C_{jklm}u^m = 0$, where $u^k u_k \neq 0$, then $u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm} = 0$ (see [14] page 128). In particular, a contraction with u^i gives $C_{jklm} = 0$. As a consequence we may state:

Proposition 4.1. *In $n = 4$, a GRW manifold with $\nabla^m C_{jklm} = 0$ is a Robertson-Walker space-time.*

Proof. In a GRW the condition $\nabla^m C_{jklm} = 0$ is equivalent to $C_{jklm}X^m = 0$. Then, in $n = 4$, it is $C_{jklm} = 0$. \square

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